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Symbolic Solutions of the Position-Time Relations for the Conic Motion of Space Dynamics

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ABSTRACT. In this paper, symbolic computational expressions are established for the solutions of the position-time relations of the different conic motion (elliptic, parabolic and hyperbolic) of space dynamics. These expressions are obtained using the symbolic manipulation capability of the software package Mathematica.

1. Introduction

Undoubtedly true that, numerical methods used to evaluate functions of space dynamics provide very accurate motion predictions. But certainly, if full analytical formulae are utilized with nowadays existing symbols used for manipulating digital computer programs, they definitely become invaluable for obtaining digital computer programs, they definitely become invaluable for obtaining predicting with any desired accuracy. Moreover, symbolic computing algorithms for space dynamics problems represent new branch that may be called the algorithmization of space dynamics (Brumberg, 1995 and Sharaf and Saad, 1997). On the other hand, the solution of position-time relations (Kepler's equation and its variants) for the different conic motion (elliptic, parabolic and hyperbolic) appears in most applications of space dynamics (*e.g.* Vinti, 1998).

Coping with the line of the recent approach for algorithmization of space dynamics and also due to the important role of the position-time relations, the present paper is devoted to establish symbolic solutions of these relations.

2. Basic Formulations

2.1. Lagrange Expansion Theorem

Lagrange's approach to solving Kepler's equation in 1770 led to a generally useful expansion theorem given as: Consider the functional equation

$$y = x + \alpha\phi(y) \tag{2.1}$$

where α is to be considered a small parameter. Then y could be expanded in terms of x and α (Smart, 1953) as

$$y = x + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^{n-1}}{d\,x^{n-1}} [\phi(x)]^n \tag{2.2}$$

Lagrange's series is, of course, the Taylor series representation of the root of Equation (2.1). Sufficient conditions for a unique root are obtained by a direct application of Rouche's theorem for analytic function of a complex variable (Battin, 1999).

2.2. Relations between Position and Time

The relations between position and time for different conic motion are (*e.g.* Danby, 1988).

$$M = E - e\ \sin\ E\ ;\ e < 1 \qquad \textit{For Elliptic Orbits,} \tag{2.3}$$

$$M = \tan^3 \frac{1}{2} f + 3\ \tan\ \frac{1}{2}\ f\ ;\ e = 1 \qquad \textit{For Parabolic Orbits,} \tag{2.4}$$

$$M = e\ \sinh\ H - H\ ;\ e > 1 \qquad \textit{For Hyperbolic Orbits.} \tag{2.5}$$

Equation (2.3) is known as Kepler's equation, Equation (2.4) as Barker's equation, while Equation (2.5) is the hyperbolic form of Kepler's equation. The angle f is the true anomaly, E and H are respectively, the elliptic eccentric anomaly and the hyperbolic eccentric anomaly. The mean anomaly M is related to the time t for the respect orbits by

$$M = \sqrt{\mu / a^3}\ (t - \tau), \tag{2.6}$$

$$M = 6\sqrt{\frac{\mu}{p3}}\ (t - \tau), \tag{2.7}$$

$$M = \sqrt{\frac{\mu}{(-a)^3}}\ (t - \tau), \tag{2.8}$$

where μ is the gravitational parameter (universal gravitational constant times the mass of the central attracting body), τ is the time of pericenter passage, while a and p are respectively, the semi-major axis and the semi-latus rectum of the orbit, finally, e is the eccentricity of the orbit.

3. Symbolic Expressions

● Consider Kepler's Equation (2.3) as

$$E = M + e \sin E.$$

Comparing this equation with Equation (2.1) we get

$$y \equiv E \,;\, x \equiv M \,;\, \alpha = e \,;\, \phi(y) \equiv \sin y$$

Then applying Lagrange's expansion theorem as given by Equation (2.2) we obtain

$$E = M + \sum_{n=1}^{\infty} \frac{e^n}{n!} \frac{d^{n-1}}{dM^{n-1}} (\sin M)^2 \tag{3.1}$$

Equation (3.1) could be written as

$$E = M + \sum_{i=1} C_i \sin iM \tag{3.2}$$

Or

$$E = M + \sum_{i=1} W_i \, e^i \,, \tag{3.3}$$

where C's are functions of e, while W's are functions of sin M. Symbolic expressions of these coefficients are listed in Tables I1, I2.

TABLE I1. Symbolic expressions for $Cj, j = 1,2, \ldots, 15$ of equation (3.9).

$$C_1 = -\frac{e^{15}}{3329438515200} + \frac{e^{13}}{14863564800} - \frac{e^{11}}{88473600} + \frac{e^9}{737280} - \frac{e^7}{9216} + \frac{e^5}{192} - \frac{e^3}{8} + e$$

$$C_2 = \frac{e^{14}}{29030400} - \frac{e^{12}}{604800} + \frac{e^{10}}{17280} - \frac{e^8}{720} + \frac{e^6}{48} - \frac{e^4}{6} + \frac{e^2}{2}$$

$$C_3 = \frac{6561\ e^{15}}{5872025600} - \frac{19683\ e^{13}}{734003200} + \frac{2187\ e^{11}}{4587520} - \frac{243\ e^9}{40960} + \frac{243\ e^7}{5120} - \frac{27\ e^5}{128} + \frac{3\ e^3}{8}$$

$$C_4 = -\frac{8\ e^{14}}{42525} + \frac{2\ e^{12}}{945} - \frac{16\ e^{10}}{945} + \frac{4\ e^8}{45} - \frac{4\ e^6}{15} + \frac{e^4}{3}$$

$$C_5 = -\frac{48828125\ e^{15}}{57076088832} + \frac{48828125\ e^{13}}{7134511104} - \frac{1953125\ e^{11}}{49545216} + \frac{78125\ e^9}{516096} - \frac{3125\ e^7}{9216} + \frac{125\ e^5}{384}$$

$$C_6 = \frac{6561\ e^{14}}{358400} - \frac{729\ e^{12}}{8960} + \frac{2187\ e^{10}}{8960} - \frac{243\ e^8}{560} + \frac{27\ e^6}{80}$$

$$C_7 = \frac{96889010407\ e^{15}}{2242274918400} - \frac{1977326743\ e^{13}}{12740198400} + \frac{40353607\ e^{11}}{106168320} - \frac{823543\ e^9}{1474560} + \frac{16807\ e^7}{46080}$$

TABLE I1. Contd.

$$C_8 = -\frac{131072\ e^{14}}{467775} + \frac{8192\ e^{12}}{14175} - \frac{2048\ e^{10}}{2835} + \frac{128\ e^{8}}{315}$$

$$C_9 = -\frac{31381059609\ e^{15}}{64592281600} + \frac{3486784401\ e^{13}}{4037017600} - \frac{43046721\ e^{11}}{45875200} + \frac{531441\ e^{9}}{1146880}$$

$$C_{10} = \frac{48828125\ e^{14}}{38320128} - \frac{1953125\ e^{12}}{1596672} + \frac{78125\ e^{10}}{145152}$$

$$C_{11} = \frac{34522712143931\ e^{15}}{18549728870400} - \frac{285311670611\ e^{13}}{178362777600} + \frac{2357947691\ e^{11}}{3715891200}$$

$$C_{12} = \frac{1458\ e^{12}}{1925} - \frac{52488\ e^{14}}{25025}$$

$$C_{13} = \frac{1792160394037\ e^{13}}{1961990553600} - \frac{302875106592253\ e^{15}}{109871471001600}$$

$$C_{14} = \frac{1977326743\ e^{14}}{1779148800}$$

$$C_{15} = \frac{320361328125\ e^{15}}{235115905024}$$

TABLE I2. Symbolic expressions for $Wj, j = 1,2, \ldots, 15$ of equation (3.3).

$$W_1 = \sin(M)$$

$$W_2 = \frac{1}{2}\sin(2M)$$

$$W_3 = \frac{3}{8}\sin(3M) - \frac{\sin(M)}{8}$$

$$W_4 = \frac{1}{3}\sin(4M) - \frac{1}{6}\sin(2M)$$

$$W_5 = \frac{\sin(M)}{192} - \frac{27}{128}\sin(3M) + \frac{125}{384}\sin(5M)$$

$$W_6 = \frac{1}{48}\sin(2M) - \frac{4}{15}\sin(4M) + \frac{27}{80}\sin(6M)$$

$$W_7 = -\frac{\sin(M)}{9216} + \frac{243\ \sin(3M)}{5120} - \frac{3125\ \sin(5M)}{9216} + \frac{16807\ \sin(7M)}{46080}$$

$$W_8 = -\frac{1}{720}\sin(2M) + \frac{4}{45}\sin(4M) - \frac{243}{560}\sin(6M) + \frac{128}{315}\sin(8M)$$

TABLE I2. Contd.

$$W_9=\frac{\sin(M)}{737280}-\frac{243\sin(3M)}{40960}+\frac{78125\sin(5M)}{516096}-\frac{823543\sin(7M)}{1474560}+\frac{531441\sin(9M)}{1146880}$$

$$W_{10}=\frac{\sin(2M)}{17280}-\frac{16}{945}\sin(4M)+\frac{2187\sin(6M)}{8960}-\frac{2048\sin(8M)}{2835}+\frac{78125\sin(10M)}{145152}$$

$$W_{11}=-\frac{\sin(M)}{88473600}+\frac{2187\sin(3M)}{4587520}-\frac{1953125\sin(5M)}{49545216}+\frac{40353607\sin(7M)}{106168320}-\frac{43046721\sin(9M)}{45875200}+\frac{2357947691\sin(11M)}{3715891200}$$

$$W_{12}=\frac{\sin(2M)}{604800}+\frac{2}{945}\sin(4M)-\frac{729\sin(6M)}{8960}+\frac{8192\sin(8M)}{14175}-\frac{1953125\sin(10M)}{1596672}+\frac{1458\sin(12M)}{1925}$$

$$W_{13}=\frac{\sin(M)}{14863564800}-\frac{19683\sin(3M)}{734003200}+\frac{48828125\sin(5M)}{7134511104}-\frac{1977326743\sin(7M)}{12740198400}+\frac{3486784401\sin(9M)}{4037017600}-\frac{285311670611\sin(11M)}{178362777600}+\frac{1792160394037\sin(13M)}{1961990553600}$$

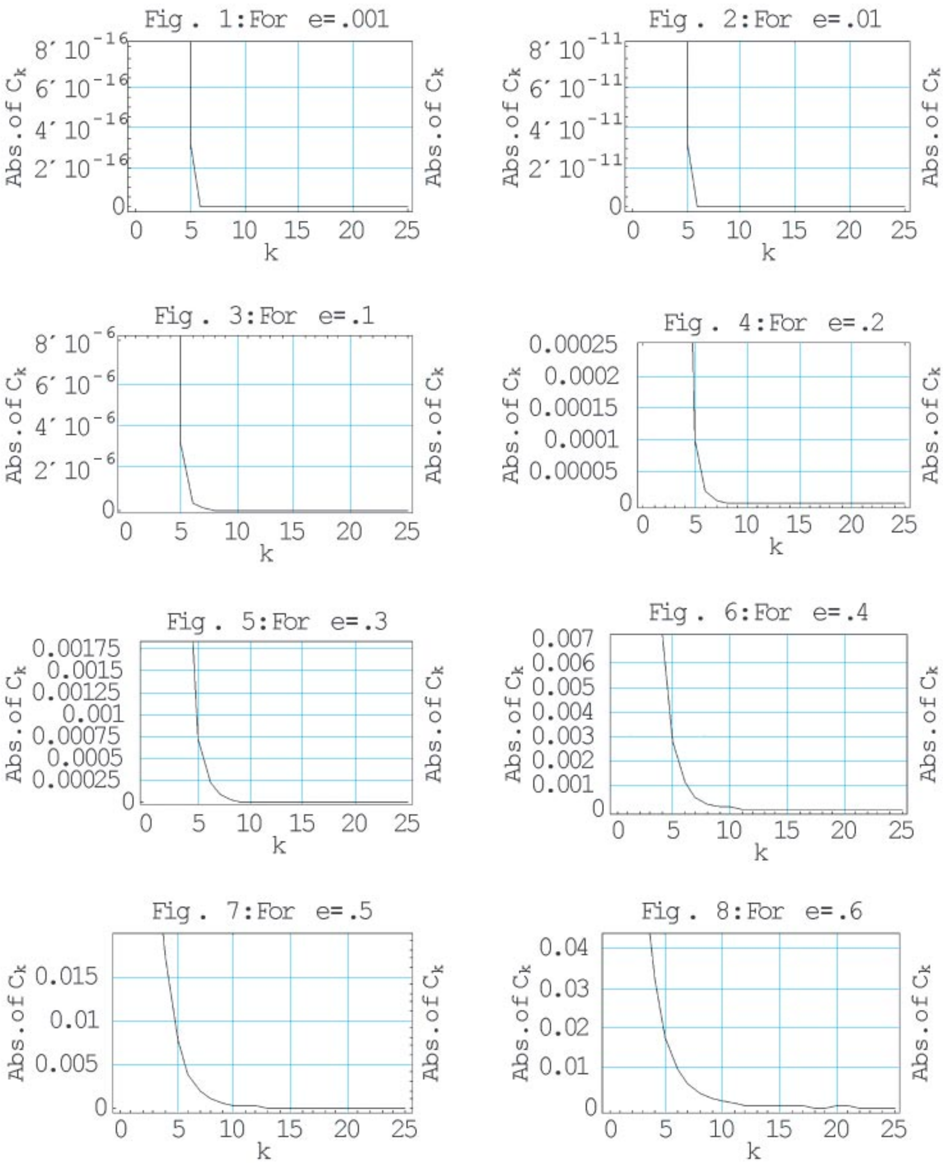
$$W_{14}=\frac{\sin(2M)}{29030400}-\frac{8\sin(4M)}{42525}+\frac{6561\sin(6M)}{358400}-\frac{131072\sin(8M)}{467775}+\frac{48828125\sin(10M)}{38320128}-\frac{52488\sin(12M)}{25025}+\frac{1977326743\sin(14M)}{1779148800}$$

$$W_{15}=\frac{\sin(2M)}{3329438515200}+\frac{6561\sin(3M)}{5872025600}-\frac{48828125\sin(5M)}{57076088832}$$
$$\frac{96889010407\sin(7M)}{2242274918400}-\frac{31381059609\sin(9M)}{64592281600}+\frac{34522712143931\sin(11M)}{18549728870400}-\frac{302875106592253\sin(13M)}{109871471001600}+\frac{320361328125\sin(15M)}{235115905024}$$

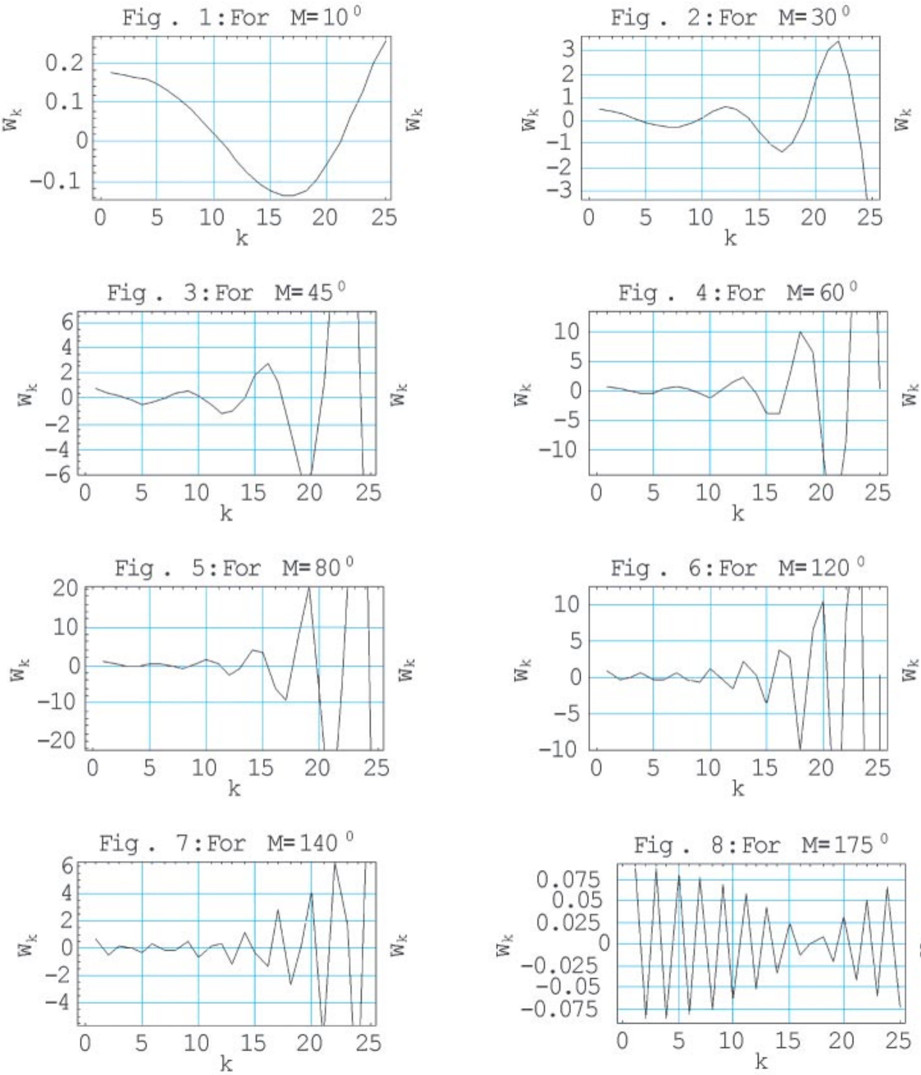
Numerical values of these coefficients are shown in Figures C and W for some e's and M's.

It could be shown (Taff, 1985) and $e < 0.66274341$ ... is the requirement for Lagrange's series (3.2) to represent the unique root of Kepler's equation for all values of the mean anomaly M. Laplace was the first to show that if e exceeds this critical value, then the series will diverge for some values of M.

● According to the classical formula of Cardan, Barker's Equation (2.4) has exact solution (Battin, 1999) of the form



FIGS. C. Graphical representations of the absolute values of the *C*'s coefficients for some values of *e*.



FIGS. W. Graphical representations of the numerical values of the *W*'s coefficients for some values of *M*.

$$y = \tan\frac{1}{2}f = \left(B+\sqrt{B^2+1}\right)^{\frac{1}{3}} - \left(B+\sqrt{B^2+1}\right)^{\frac{-1}{3}} \tag{3.4}$$

Where

$$B = \frac{1}{2}M \tag{3.5}$$

From Equation (2.4) and Descarte's rule of changing signs it is clear that, one and only one real root exists for this equation.

Using binomial theorem, Equation (3.4) could be represented as

$$y = \sum_{j=1} S_j \; B^{2j-1} , \tag{3.6}$$

where the numerical values of the *S'*s coefficients are listed in Table II and displayed graphically in Figure S.

TABLE II. Numerical values of the S_j , j = 1, 2, ... , 44 coefficients of equation (3.6).

S_1 = 0.666667	S_2 = –0.0987654	S_3 = 0.0438957	S_4 = – 0.0260123
S_5 = 0.0176627	S_6 = –0.0129883	S_7 = 0.010065	S_8 = – 0.00809461
S_9 = 0.0066926	S_{10} = –0.00565327	S_{11} = 0.00485763	S_{12} = –0.00423256
S_{13} = 0.00373092	S_{14} = –0.0033211	S_{15} = 0.00298117	S_{16} = –0.002695522
S_{17} = 0.00245274	S_{18} = –0.00224434	S_{19} = 0.00206386	S_{20} = –0.00190634
S_{21} = 0.00176789	S_{22} = –0.00164542	S_{23} = 0.00153646	S_{24} = – 0.00143902
S_{25} = 0.00135146	S_{26} = –0.00127243	S_{27} = 0.00120082	S_{28} = –0.00113568
S_{29} = 0.00107622	S_{30} = –0.00102178	S_{31} = 0.000971777	S_{32} = –0.000925723
S_{33} = 0.000883195	S_{34} = –0.000843827	S_{35} = 0.000807298	S_{36} = –0.000773332
S_{37} = 0.000741682	S_{38} = –0.000712133	S_{395} = 0.000684496	S_{40} = –0.000658601
S_{41} = 0.000634299	S_{42} = –0.000611455	S_{43} = 0.00058995	S_{44} = –0.000569677

- Consider the hyperbolic form of Kepler's Equation (2.5) as

$$\sinh H = \frac{M}{e} + \frac{1}{e}H. \tag{3.7}$$

Comparing Equations (3.7) and (2.1) we have;

$$y \equiv \sinh H ; x = \frac{M}{e} ; \alpha = \frac{1}{e} ; \phi(y) \equiv \sinh^{-1} y,$$

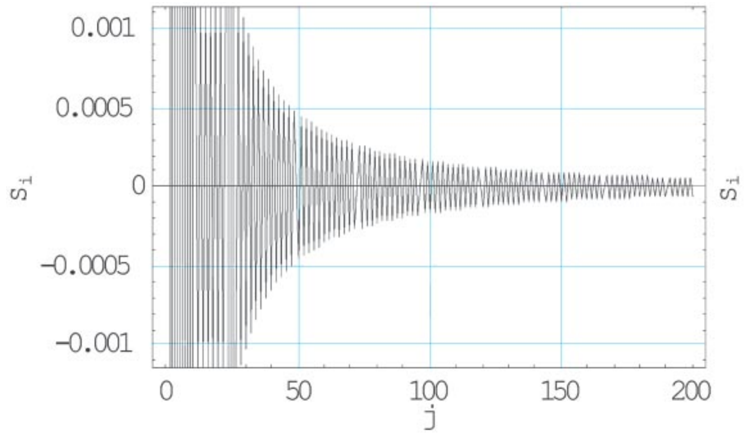


FIG. S. Graphical representations of the *S*'s coefficients.

then applying Lagrange's expansion theorem of Equation (2.2) we get

$$\sinh H = \frac{M}{e} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{1}{n!}\right)^n \frac{d^{n-1}}{d\ x^{n-1}}\left(\sinh^{-1}\ x\right)^n \ . \tag{3.8}$$

Equation (3.8) could be written as

$$H = \sum_{i=1} G_i Z^i \tag{3.9}$$

Where

$$Z = \sinh^{-1}\left(\frac{M}{e}\right), \tag{3.10}$$

G's are function of *e*, *M* and listed in Table III. Numerical values of these coefficients are displayed graphically in Figures G for some *e*'s and *M*'s. Equations (3.2) [or (3.3)], (3.6) and (3.9) are the required forms of the symbolic expressions for the solutions of Kepler's equation and its variants. Using the symbolic manipulation capability of the software package of Mathematica, we generate the coefficients of these equations and are listed in Tables I.1, I.2, II and III, while their numerical values are displayed graphically in Figures C, W, S and G.

References

[1] **Battin, R.H.** (1999) *An Introduction to the Mathematics and Methods of Astrodynamics, Revised Edition,* AIAA, Education Series, New York.

[2] **Brumberg, V.A.** (1995) *Analytical Techniques of Celestial Mechanics,* Springer-Verlag, Berlin, Heidelberg, New York.

[3] **Danby, J.M.A.** (1988) *Fundamental of Celestial Mechanics,* 2nd edn, William-Bell, Inc., Richmond, Virginia, USA.

[4] **Sharaf, M.A.** and **Saad, A.S.** (1997) Analytical Expansion of the Earth's Zonal Potential in Terms of KS Regular Elements. *Celestial Mechanics and Dynamical Astronomy,* **66:** 181.
[5] **Smart, W.M.** (1953) *Celestial Mechanics,* Longmans and Green, London.
[6] **Taff, G.L.** (1985) *Celestial Mechanics: A Computational Guide for the Practitioner,* John Wiley and Sons, Inc., USA.
[7] **Vinti, J.P.** (1998) *Orbital and Celestial Mechanics,* Progress in Astronautics and Aeronautics, Vol. **177,** AIAA, USA.

TABLE III. Symbolic expressions for G_j , j = 1,2, ... , 11 of equation (3.9).

$$G_1 = \frac{e\sqrt{\frac{M^2}{e^2}+1}\,\left((e^2+M^2)^6+(e^2+M^2)^5+(e^2+M^2)^4+(e^2+M^2)^3+(e^2+M^2)^2+e^2+M^2+1\right)+}{(e^2+M^2)^7}$$

$$\frac{1}{e^2+M^2}+\frac{1}{(e^2+M^2)^2}+\frac{1}{(e^2+M^2)^3}+\frac{1}{(e^2+M^2)^4}+\frac{1}{(e^2+M^2)^5}+\frac{1}{(e^2+M^2)^6}+\frac{1}{(e^2+M^2)^7}+1$$

$$G_2 = \frac{1}{2\,(e^2+M^2)^8}\left(M\left(e\sqrt{\frac{M^2}{e^2}+1}\,(-(e^2+M^2)^6-6(e^2+M^2)^5-15(e^2+M^2)^4-28(e^2+M^2)^3-\right.\right.$$

$$45\,(e^2+M^2)^2-66\,(e^2+M^2)-91)-(e^2+M^2)$$

$$\left.\left.(3\,(e^2+M^2)^5+10\,(e^2+M^2)^4+21\,(e^2+M^2)^3+36\,(e^2+M^2)^2+55\,(e^2+M^2)+78)\right)\right)$$

$$G_3 = \frac{1}{6\,(e^2+M^2)^8}$$

$$\left(3\,M^2\left(e\sqrt{\frac{M^2}{e_2}+1}\,(e^2+M^2)^5+5\,(e^2+M^2)^5+15\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^4+35\,(e^2+M^2)^4+\right.\right.$$

$$70\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^3+126\,(e^2+M^2)^3+210\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^2+330$$

$$\left.(e^2+M^2)^2+495\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)+715\,(e^2+M^2)+1001\,e\sqrt{\frac{M^2}{e^2}+1}+1365\right)-$$

$$(e^2+M^2)\left(e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^5+4\,(e^2+M^2)^5+10\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^4+20\right.$$

$$(e^2+M^2)^4+35\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^3+56\,(e^2+M^2)^3+84\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)^2+$$

$$\left.\left.120\,(e^2+M^2)^2+165\,e\sqrt{\frac{M^2}{e^2}+1}\,(e^2+M^2)+220\,(e^2+M^2)+286\,e\sqrt{\frac{M^2}{e^2}+1}+364\right)\right)$$

TABLE III. Contd.

$$G_4 = \Big(M\Big(9\,e^{12} + 39\,M^2\,e^{10} + 195\,e^{10} + 60\,M^4\,e^8 + 555\,M^2\,e^8 + 1190\,e^8 + 30\,M^6 e^6 + 270\,M^4\,e^6 + 1610$$

$$M^2\,e^6 + 4410\,e^6 - 15\,M^8\,e^4 - 570\,M^6\,e^6 - 2310\,M^4\,e^4 - 630\,M^2\,e^4 + 12375\,e^4 -$$

$$21\,M^{10}\,e^2 - 705\,M^8\,e^2 - 4690\,M^6\,e^2 - 14490\,M^4\,e^2 - 20295\,M^2\,e^2 + 29029\,e^2 +$$

$$\sqrt{\frac{M^2}{e^2}+1}\;(-50\,M^{10} - 5\,(29\,e^2 + 147)\,M^8 - 8\,(10\,(e^2+21)\,e^2 + 567)\,M^6 + 2\,(65\,e^6 -$$

$$315\,e^4 - 3339\,e^2 - 9075)\,M^4 + 2\,(85\,e^8 + 420\,e^6 + 126\,e^4 - 5280\,e^2 - 27885)\,M^2 +$$

$$e^2\,(55\,e^8 + 525\,e^6 + 2394\,e^4 + 7590\,e^2 + 19305))\,e - 6\,M^{12} - 225\,M^{10} -$$

$$1960\,M^8 - 9450\,M^6 - 32670\,M^4 - 91091\,M^2\Big)\Big)\Big/\left(24\,e\,(e^2+M^2)^8\sqrt{\frac{M^2}{e^2}+1}\right)$$

$$G_5 = \frac{1}{120\,(e^2+M^2)^9}$$

$$\Big(64\,e^{12} - 351\,M^2\,e^{10} + 784\,e^{10} - 1770\,M^4\,e^8 - 7420\,M^4\,e^8 + 4368\,e^8 - 2290\,M^6\,e^6 -$$

$$20195\,M^4\,e^6 - 58758\,M^2\,e^6 + 16368\,e^6 - 720\,M^8\,e^4 - 8225\,M^6\,e^4 - 67347\,M^4\,e^4 -$$

$$285516\,M^2\,e^4 + 48048\,e^4 + 489\,M^{10}\,e^2 + 10535\,M^8\,e^2 + 59052\,M^6\,e^2 +$$

$$55539\,M^4\,e^2 - 1030029\,M^2\,e^2 + \sqrt{\frac{M^2}{e^2}+1}\;(9\,e^{12} + (259 - 36\,M^2)e^{10} -$$

$$7\,(30\,M^4 + 295\,M^2 - 282)\,e^8 - (300\,M^6 + 6125\,M^4 + 23604\,M^2 - 8778)\,e^6 -$$

$$(135\,M^8 + 3395\,M^6 + 30681\,M^4 + 139986\,M^2 - 28743)\,e^4 +$$

$$M^2(24\,M^8 + 2030\,M^6 + 17346\,M^4 + 9009\,M^2 - 573144)\,e^2 +$$

$$M^4(24\,M^8 + 1624\,M^6 + 22449\,M^4 + 157773\,M^2 + 749463))\,e +$$

$$274\,M^{12} + 6769\,M^{10} + 63273\,M^8 + 357423\,M^6 + 1474473\,M^4\Big)$$

$$G_6 = \Big(M\Big(-225\,e^{12} - 300\,M^2\,e^{10} - 10612\,e^{10} + 930\,M^4\,e^8 + 6790\,M^2\,e^8 -$$

$$120330\,e^8 + 2220\,M^6\,e^6 + 70910\,M^4\,e^6 + 315630\,M^2\,e^6 - 745668\,e^6 +$$

$$1455\,M^8\,e^4 + 65870\,M^6\,e^4 + 722925\,M^4\,e^4 + 3438666\,M^2\,e^4 - 3246243\,e^4 +$$

$$120\,M^{10}\,e^2 - 770\,M^8\,e^2 + 17640\,M^6\,e^2 + 1546776\,M^4\,e^2 + 21435414\,M^2\,e^2 -$$

$$3\sqrt{\frac{M^2}{e^2}+1}\;(693\,e^{10} - 21\,(5\,M^2 - 628)\,e^8 - 7\,(555\,M^4 + 4014\,M^2 - 15070)\,e^6 -$$

TABLE III. Contd.

$$3(7\, M^2\,(195\, M^4+3492\, M^2+20680)-179036)\, e^4-$$

$$3\, M^2\,(7\,(20\, M^4+458\, M^2+11385)\, M^2+1093378)\, e^2+$$

$$M^4\,(7\,(84\, M^4+3204\, M^2+42955)\, M^2+2308878))\, e-120M^{12}-13132M^{10}-$$

$$269325\, M^8-2637558\, M^6-16669653\, M^4))\bigg/\left(720\, e\,(e^2+M^2)^9\sqrt{\frac{M^2}{e^2}+1}\right)$$

$$G_7=-\frac{1}{5040\,(e^2+M^2)^{10}}\Big(\,2304\, e^{12}-39663\, M^2\, e^{10}+52480\, e^{10}-59625M^4\, e^8-$$

$$1189435\, M^2\, e^8+489280\, e^8+69075\, M^6\, e^6+122860\, M^4\, e^6-13763585\, M^2\, e^6+$$

$$2846272\, e^6+133785\, M^8\, e^4+3300265\, M^6\, e^4+21989715\, M^4\, e^4-95650269\, M^2\, e^4+$$

$$33984\, M^{10}\, e^2+1211810\, M^8\, e^2+22903045\, M^6\, e^2+290250246\, M^4\, e^2+$$

$$\sqrt{\frac{M^2}{e^2}+1}\;(225\, e^{12}+(12916-3375\, M^2)\, e^{10}-5\,(1215\, M^4+52793\, M^2-34562)\, e^8+$$

$$(3555\, M^6-41570\, M^4-4475075\, M^2+1234948)\, e^6+$$

$$5\, M^2\,(1998\, M^6+128459\, M^4+1206678\, M^2-7747311)\, e^4+$$

$$M^4\,(3240\, M^6+288860\, M^4+7264345\, M^2+108011046)\, e^2-$$

$$M^6\,(720\, M^6+118124\, M^4+3416930\, M^2+44990231)\, e-$$

$$13068\, M^{12}-723680\, M^{10}-13339535\, M^8-135036473\, M^6$$

$$G_8=\Big(M\Big(11025\, e^{10}-33075\, M^2\, e^{10}+909765\, e^{10}-112455\, M^4\, e^8-4966065\, M^2\, e^8+16571775\, e^{10}-$$

$$33705\, M^6\, e^6-5313285\, M^4\, e^6-131415405\, M^2\, e^6+154821667\, e^6+77490\, M^8\, e^4+$$

$$6737985\, M^6\, e^4+54971235\, M^4\, e^4-1611568959\, M^2\, e^4+37800\, M^{10}\, e^2+$$

$$5002740\, M^8\, e^2+156962685\, M^6\, e^2+2680498821\, M^4\, e^2+\sqrt{\frac{M^2}{e^2}+1}$$

$$(136431\, e^{10}-5\,(128988\, M^2-867647)\, e^8-5\,(167193\, M^4+6241114\, M^2-10776909)$$

$$e^6+15\;\; M^2\,(46602\, M^4+537097\, M^2-34733556)\, e^4+5\, M^4\,(128808\, M^4+$$

$$7038152\, M^2+158078349)\, e^2-2\, M^6(54792\, M^2+4204750\, M^2+103035075))\;\; e-$$

$$5040\, M^{12}-1172700\, M^{10}-45995730\, M^8-790943153\, M^6\,\Big)\Big)$$

$$\left(40320\, e(e^2+M^2)^{10}\sqrt{\frac{M^2}{e^2}+1}\right)$$

TABLE III. Contd.

$$G_9 = \frac{1}{362880\,(e^2+M^2)^{11}}$$

$$\big(147456\, e^{12} - 4857759\, M^2 e^{10} + 5395456\, e^{10} + 7147980\, M^4 e^8 - 215051650\, M^2 e^8 -$$

$$75851776\, e^8 + 18632205\, M^6 e^6 + 679756165\, M^4 e^6 - 3549516971\, M^2 e^6 -$$

$$3321270\, M^8 e^4 + 167319955\, M^6 e^4 + 17091805731\, M^6 e^4 -$$

$$8773704\, M^{10} e^2 - 627625240\, M^8 e^2 - 17194638461\, M^6 e^2 + \sqrt{\frac{M^2}{e^2}+1}$$

$$(11025\, e^2 + (1057221 - 330750\, M^2)\, e^{10} + 11\,(225\,(147\, M^2 - 15761)\, M^2 + 1997021)\, e^8 +$$

$$(1199520\, M^6 + 108979200\, M^4 - 962700596\, M^2)\, e^6 +$$

$$(1199520\, M^6 + 108979200\, M^4 - 962700596\, M^2)\, e^6 +$$

$$6\, M^4 (30\,(232397 - 168\, M^2)\, M^2 + 713798371)\, e^4 - 4\, M^6 (45\,(2688\, M^2 + 524777)\, M^2 +$$

$$974067809)\, e^2 + 4\, M^8 (10080\, M^4 + 3188394\, M^2 + 164301709))\;\; e\; +$$

$$1026576\, M^{12} + 105258076\, M^{10} + 3336118786\, M^8\big)$$

$$G_{10} = \Big(M\big(-893025\, e^{12} + 162\,(47775\, M^2 - 704143)\, e^{10} +$$

$$(45\, M^2\,(22491\, M^2 + 29431490) - 3049587541)\, e^8 -$$

$$4\, M^2\,(45\, M^2\,(101304\, M^2 + 9804751) - 11107455359)\, e^6 -$$

$$6\, M^4\,(180\,(4116\, M^2 + 1447105)\, M^2 + 19698159481)\, e^4 +$$

$$4\, M^6\,(45\,(32256\, M^2 + 8275289)\, M^2 + 19111156064)\, e^2 -$$

$$33\sqrt{\frac{M^2}{e^2}+1}\;\; 419775\, e^{10} - 3\,(1488275\, M^2 - 6695702)\, e^8 +$$

$$4\, M^2\,(1236345\, M^2 - 68237494)\, e^6 + 12\, M^4\,(453845\, M^2 + 55565393)\, e^4 -$$

$$12\, M^6 (338480\, M^2 + 32179043)\, e^2 + 8\, M^8 (40260\, M^2 + 5356117))\;\; e -$$

$$4\, M^8\,(90720\, M^4 + 37729494\, M^2 + 2489425939)\big)\Big) \Big/ \left(3628800\, e\,(e^2+M^2)^{11}\sqrt{\frac{M^2}{e^2}+1}\right)$$

$$G_{11} = \left(-\,e\,\sqrt{\frac{M^2}{e^2}+1}\;\; (-40181280\, M^{12} +\right.$$

$$8\,(80726220\, e^2 - 846406379)\, M^{10} - 60\, e^2\,(22812789\, e^2 - 1279180357)\, M^8 -$$

$$480\, e^4\,(1086061\, e^2 + 369560828)\, M^6 + 35\, e^6\,(36275445\, e^2 + 3120656812)\, M^4 -$$

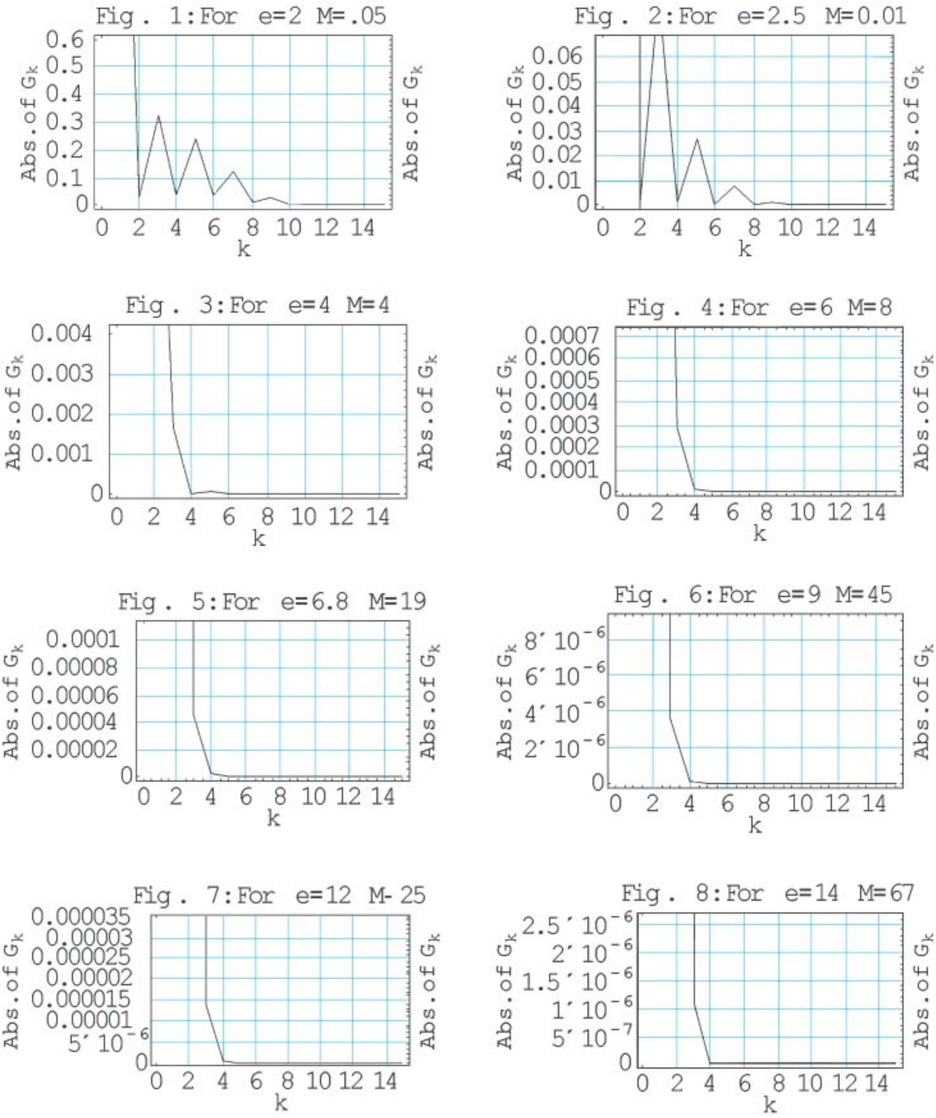
$$225\, e^8\,(1150637\, e^2 + 71601075)\, M^2 + 49152\, e^{10}\,(100\, e^2 + 5369)) - 3(e^2 + M^2)$$

$$(99225\, e^{12} + (14312974 - 4862025\, M^2)\, e^{10} + 13\, M^2\,(1653750\, M^2 + 63147979)\, e^8 +$$

$$(5167761352\, M^4 - 5292000\, M^6)\, e^6 - 8\, M^6\,(2835000\, M^2 + 956959159)\, e^4 +$$

$$128\, M^8\,(67725\, M^2 + 22901749)\, e^2 - 128\, M^{10}\,(3150\, M^2 + 1676701))\big)$$

$$\left(13305600\;\; e\;(e^2 + M^2)^{12}\sqrt{\frac{M^2}{e^2}+1}\right)$$



FIGS. G. Graphical representations of the absolute values of the *G's* coefficients for some values of *e* and *M.*

الحلول الرمزية لعلاقات الموضع والزمن في الحركة المخروطية لديناميكا الفضاء

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المستخلص. تم في هذا البحث تشييد صيغ حسابية رمزية لحل علاقات الموضع - الزمن وذلك لأنواع الحركة المخروطية المختلفة « ناقصية ، مكافئة ، زائدية » في ديناميكا الفضاء .

استخدم برنامج (Mathematica) مثمتكا لقدرته على المعالجة الرمزية في الحصول على هذه الصيغ.