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ON SOME ASYMPTOTIC BEHAVIOUR OF ABSTRACT MEASURE SYSTEMS

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Abstract

Existence and uniqueness of solutions of a system of abstract measure of differential equations are investigated. Local stability is studied. The main tools are contraction mapping principle and Schauder's fixed point theorem.

1. Introduction

Let R denote the real line, R^n denote the Euclidean space with respect to the norm $|\cdot|_m$ defined by

$$|x|_m = \max(|x_1|, |x_2|, \ldots, |x_n|)$$

and let X be a Banach space with norm denoted by $\|\cdot\|$. For any two points $x, y \in X$, the segment $\overline{xy}$ is defined by

$$\overline{xy} = \{z \in X : z = x + \lambda(y - x), 0 \leq \lambda < 1\}.$$

Let x_0, y_0 be two fixed points of X and z be a variable point of X such that $\overline{x_0 z}$ and $\overline{y_0 z}$ are nonempty and $\overline{x_0 z} \subset \overline{y_0 z}$. For $x_1, x_2 \in \overline{y_0 z}$, we

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write $x_1 < x_2$ if $\overline{y_0x_1} \in \overline{y_0x_2}$. For any point $x \in \overline{y_0z}$, we define the sets S_x and $\overline{S}_x$ as follows:

$$S_x = \{\lambda x : -\infty < \lambda < 1\}, \quad \overline{S}_x = \{\lambda x : -\infty < \lambda \leq 1\}.$$

Let $w = \| x_0 - y_0 \|$ denote the distance between x_0 and y_0. For each $x \in \overline{x_0z}$, there exists a unique vector $x' < x$ such that $x' < x$ and $\| x - x' \| = w$, and we denote this vector by xw. Note that xw and wx are identical if and only if $w = 0$ and $x = 0$ (the zero vector of X).

By a vector measure p defined on a σ-algebra M, we mean an ordered n-tuple $(p_1, p_2, ..., p_n)$ of n real measures. The norm $\| p \|_n$ of p is defined by

$$\| p \|_n = \max(\| p_1 \|, \| p_2 \|, ..., \| p_n \|),$$

where $\| p_i \|$, $i = 1, 2, ..., n$, denotes the usual norm of the real measure p_i defined in [2]. Let $ca(X, M)$ be the space of all vector measures defined on M. It is clear that $ca(X, M)$ is a Banach space with respect to the norm defined above. If μ is a positive measure on M and $p \in ca(X, M)$, then we say that p is absolutely continuous with respect to μ, if $\mu(E) = 0$ implies $p(E) = 0$ (the zero vector in R^n). In this case we write $p << \mu$. For $p \in ca(X, M)$, we define a positive measure $| p |_n$ by

$$| p |_m(E) = \max(| p_1 |(E), ..., | p_n |(E)),$$

where $| p - i |$ denotes the total variation measure of the real measure p_i as defined in [2].

We shall use the concept available in the following theorem:

Theorem A [2]. *Let μ be a positive measure on a σ-algebra M. Then*

(a) $\mu(\varnothing) = 0$.

(b) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots \mu(A_n)$; *if* $A_1, ..., A_n$ *are positive disjoint members of M.*

(c) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$ *if* $A \in M$, $B \in M$.

(d) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in M$ *and* $A_1 \subset A_2 \subset A_3 \subset \dots$.

(e) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in M$ *and* $A_1 \supset A_2 \supset A_3 \supset \dots$ *and* $\mu(A)$ *is finite.*

Let M_0 denote the smallest σ-algebra on $\overline{S}_{x_0}$ containing x_i and the sets $\overline{S}_x$, $x \in \overline{y_0 x_0}$. For any $z > x_0$, let M_z denote the smallest σ-algebra defined on S_z containing M_0 and the sets $\overline{S}_x$, $x \in \overline{x_0 z}$. For a given positive number H, we define the sets B_H and C_H by

$$B_H = \{u : u \in R^n, |u|_m < H\}, \quad C_H = \{q \in ca(\overline{S_{x_0}}, M_0) : \| q \|_n < H\}.$$

Now we consider the following system involving the delay w

$$\frac{dp}{d\mu} = f(x, p(\overline{S_{xw}})) \tag{1.1}$$

and the initial condition

$$p(E) = q(E), \quad E \in M_0, \tag{1.2}$$

where $q \in C_H$ is a known vector measure, $\frac{dp}{d\mu}$ is the Radan-Nikodym derivative of p with respect to μ, and $f(x, y)$ is an R^n-valued function defined on $S_z \times B_H$, such that for each $p \in ca(S_z, M_z)$, $f(x, p(\overline{S_{xw}}))$ is μ-integrable.

It can be observed that the equation (1.1) is equivalent to n differential equations

$$\frac{dp_i}{d\mu} = f_i(x, p(\overline{S_{xw}})), \quad i = 1, 2, \dots, n,$$

satisfying $p_i(E) = q_i(E)$, where $p_1, p_2, \dots, p_n \in p$, $f_1, f_2, \dots, f_n$, and $q_1, q_2, \dots, q_n \in q$.

Definition 1.1. Given an initial measure $q \in C_H$, a vector measure $p \in ca(S_z, M_z)$ (for some $z > x_0$) is said to be a *solution* of (1.1), (1.2) if

(i) $p(E) = q(E)$, $E \in M_0$,

(ii) $p < \mu$ on $\overline{x_0 z}$,

(iii) $p(E) \in B_H$, $E \in M_0$,

(iv) p satisfies (1.1) a.e. $[\mu]$ on $\overline{x_0 z}$.

It is clear that the conditions (ii) and (iv) together are equivalent to the following condition

$$p(E) = \int_E f(x, p(\overline{S_{xw}}))d\mu, \quad E \in \overline{x_0 z}$$

when we say that p satisfies condition (ii) or (iv), we mean that it holds for all measurable subsets $\overline{x_0 z}$.

Definition 1.2. A solution p of (1.1) and (1.2) existing on $\overline{x_0 z}$ will be denoted by $p(\overline{S_{x_0}}, q)$.

2. Main Result

In this section we investigate the existence, uniqueness, extension and stability of solution p for the systems (1.1) and (1.2) using fixed point theorems due to Schauder and Banach [1, 2, 3], respectively.

We assume the following:

(i) $\mu(x_0) = 0$.

(ii) For any $z > x$, M_z is compact with respect to the topology generated by the metric "d" defined by

$$d(E_1, E_2) = \mu(E_1 \Delta E_2), \quad E_1, E_2 \in M_z,$$

where $E_1 \Delta E_2$ is the set symmetric difference between E_1 and E_2.

(iii) There exists a μ-integrable real function $U(x)$ defined on S_z such that

$$|f(x, y)|_m \leq U(x), \quad (x, y) \in S_z \times B_H.$$

(iv) $f(x, y)$ is continuous in y for each $x \in S_z$.

(v) q is continuous on M_z with respect to the pseudo-metric d defined in (ii).

(vi) $f(x, y)$ satisfies Lipschitz condition of the type

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

for $(x, y_1), (x, y_2) \in S_z \times B_H$, where L is Lipschitz constant.

Theorem 2.1. *Let the assumptions* (i)-(vi) *hold. Then for a given initial measure* $q \in C_H$, *there exists a solution* $p(\overline{S_{x_0}}, q)$ *of* (1.1) *and* (1.2) *on* $\overline{x_0 x_1}$ *for some* $x_1 > x_0$.

Proof. Let $r_n (> 1)$ be a decreasing sequence of real numbers, such that $r_n \to 1$ as $n \to \infty$ and

$$S_{r_1 x_0} \supset S_{r_2 x_0} \supset \cdots \supset S_{r_2 x_0},$$

then we have

$$\lim_{n \to \infty} \mu\{(\overline{S_{r_n x_0}} - \overline{S_{x_0}})\} = 0.$$

This shows that there exists a number r^* and point $x_1 = r^* x_0$ such that $\overline{S_{x_0}} \subset S_{x_1}$ and

$$\int_{\overline{x_0 x_1}} U(x) d\mu < H - \| q \|_n. \tag{2.1}$$

This is possible by virtue of (i) and positiveness of μ. Now in the Banach space $B_0 = ca(S_{x_1}, M_{x_1})$ defined by the subset

$$S = \{p \in B_0, \, p(E) = q(E) \text{ if } E \in M_0 \text{ and } \| p \|_n \leq k\},$$

where

$$k = \| p \|_n + \int_{\overline{x_0 x_1}} U(x) d\mu. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\| p \|_n < H, \; p \in S.$$

Now we define the operator T on S by

$$(Tp)(E) = q(E), \quad E \in M_0 \tag{2.3}$$

$$(Tp)(E) = \int_E f(x, p(\overline{S_{xw}})) d\mu, \quad E = \overline{x_0 x_1}. \tag{2.4}$$

Let $p \in S$ and $E \in M_{x_1}$, then there exist two disjoint sets F and G in M_{x_1} such that $E = F \cup G$, $E \in M_0$ and $G \subset \overline{x_0 x_1}$. Then from (2.3) and (2.4) it follows that

$$| (Tp)(E) |_m \leq | q |_m (F) + \int_G | f(x, p(\overline{S_{xw}})) |_m d\mu$$

which by virtue of (ii) and (2.2) implies that

$$\| (Tp) \|_n \leq \| q \|_n + \int_{\overline{x_0 x_1}} U(x) d\mu = k.$$

This shows that T maps S into itself. The continuity of $f(x, y)$ in y leads to the continuity of T. To show that T is compact on S, consider a sequence $\{p_n\}$ in TS (image of S under T). Clearly, $\{p_n\}$ is uniformly bounded. Let E_1 and E_2 by any two sets in $M - x_1$. Then as before we have

$$E_i = F_i \cup G_i, \quad F_i \in M_0, \quad G_i \subset \overline{x_0 x_1}$$

and

$$F_i \cap G_i = \varnothing, \quad i = 1, 2.$$

Now from (2.3) and (2.4) and the condition

$$G_1 = (G_1 - G_2) \cup (G_1 \cap G_2), \quad G_2 = (G_2 - G_1) \cup (G_1 \cap G_2),$$

we obtain

$$p_n(E_1) - p_n(E_2) = q(F_1) - q(F_2) + \int_{G_1 - G_2} f(x,\, p_n(\overline{S_{xw}}))d\mu$$

$$- \int_{G_2 - G_1} f(x,\, p_n(\overline{S_{xw}}))d\mu.$$

This by virtue (iii) implies that

$$| p_n(E_1) - p_n(E_2) |_m < | q(F_1) - q(F_2) |_m + \int_{G_1 \Delta G_2} U(x)d\mu. \qquad (2.5)$$

Hence from (2.5), (v) and μ-integrability of U, we conclude that

$$| p_n(E_1) - p_n(E_2) |_m \to 0 \text{ as } d(E_1,\, E_2) \to 0$$

which shows that the sequence $\{p_n\}$ is equicontinuous. By (ii) M_{x_1} is compact. Therefore, by Ascoli's theorem, we further conclude that TS is compact and hence T is a compact operator. Thus T is completely continuous on S. An application of Schauder's fixed point theorem now shows that there exists a solution $p(\overline{S_{x_0}},\, q)$ of (1.1) and (1.2) existing on $\overline{x_0 x_1}$ and this completes the proof.

Theorem 2.2. *Let the assumptions* (i), (ii) *and* (vi) *hold and* $q \in C_H$. *Then there exists a unique solution* $p(\overline{x_0},\, q)$ *of* (1.1) *and* (1.2) *existing on* $\overline{x_0 x_1}$ *for some* $x_1 > x_0$.

Proof. Construct a point x_1 as in Theorem 2.1 satisfying the additional condition

$$L.\mu(\overline{x_0 x_1}) < 1. \qquad (2.6)$$

Define the set S and the operator T on S as in Theorem 2.1. Then T maps S into itself. Also the assumption (vi) and the condition (2.6) imply that T is a contraction operator on S. Applying Banach's fixed point theorem, there is a unique solution $p(\overline{S_{x_0}}, q)$ of (1.1) and (1.2) existing on $\overline{x_0x_1}$. This completes the proof.

Under the hypotheses of Theorem 2.1, it can be shown that a solution $p(\overline{S_{x_0}}, q)$ of (1.1) and (1.2) existing on $\overline{x_0x_1}$ can be extended to larger segment, whenever $\mu\{x_1\} = 0$.

Theorem 2.3. *Under the hypotheses of Theorem* 2.1, *let* $p = p(\overline{S_{x_0}}, q)$ *be a solution of* (1.1) *on* $\overline{x_0x_1}$. *Then the solution* p *can be extended to a larger segment if* $\mu\{x_1\} = 0$.

Proof. Consider $p = p(\overline{S_{x_0}}, q)$ as the initial measure defined on M_{x_1}. Note that $\| p \|_n < H$. Also, by the assumption $\mu\{x_1\} = 0$. Hence by Theorem 2.1, there exists a solution $p_1 = p_1(\overline{S_{x_1}}, p)$ of (1.1) on $\overline{x_1x_2}$ for some $x_2 > x_1$, satisfying the initial condition $p_1(E) = p(E)$, if $E \in M_{x_1}$. This solution p_1 is defined on M_{x_1}, the smallest σ-algebra defined on S_{x_2}, containing $\{x_1\}$, M_{x_1} and the set $\overline{S}_x$, $x \in \overline{x_1x_2}$. It is clear that $p_1(E) = q(E)$, $E \in M_0$. Hence p is the desired extension of p_1.

Definition 2.1. Let $q \in C_H$. If for each $\varepsilon > 0$, $(\varepsilon < H)$ there exists a number $\eta = \eta(\varepsilon)$ and a solution $p(\overline{S_{x_0}}, q)$ of (1.1) and (1.2) such that $\| q \|_n \leq \eta$ implies $\| p \|_n < \varepsilon$, then we say that the solution p is locally stable with respect to the initial measure q.

For our next theorem we need the following assumptions:

(vii) $f(x, y)$ is a μ-integrable function on $S_z \times B_H$ and $f(x, 0) = 0$, where $z > x_0$ is fixed.

(viii) Given $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $| f(x, y_1) - f(x, y_2) |_m \leq \delta | y_1 - y_2 |_m$ whenever $| y_1 |_m, | y_2 |_m \leq \varepsilon$.

Theorem 2.4. *Let the assumptions* (vii) *and* (viii) *hold. Then there exists a number* $\varepsilon_0 > 0$, *such that for every* ε, $0 < \varepsilon < \varepsilon_0$, *and a fixed* *number* $b \in (0, 1)$, *there is a unique solution* $p(\overline{S_{x_0}}, q)$ *of* (1.1) *and* (1.2) *satisfying* $\| p \|_n < \varepsilon$, *wherever* $\| q \|_n < b\varepsilon$.

Proof. Let $\delta = \frac{1-b}{k_0}$, where $k_0 = \mu(\overline{x_0 z})$. Corresponding to this δ, there exists by (vii) a number $\varepsilon_0 > 0$, such that

$$| f(x, y_1) - f(x, y_2) |_m \leq \delta | y_1 - y_2 |_m \tag{2.7}$$

whenever $| y_1 |_m, | y_2 |_m \leq \varepsilon_0$.

Now for any ε, $0 < \varepsilon < \varepsilon_0$, define

$$S(\varepsilon) = \{p \in ca(S_z, M_z) : \| p \|_n \leq \varepsilon\}.$$

Let $p \in \overline{S}(\varepsilon)$. Then by using (vii) and (2.7), we obtain

$$|f(x, p(\overline{S_{xw}}) - f(x, 0)|_m < \delta\varepsilon. \tag{2.8}$$

Define an operator T on $\overline{S}(\varepsilon)$ by

$$(Tp)(E) = q(E), \; E \in M_0$$

$$(Tp)(E) = \int_E f(x, p(\overline{S_{xw}}))d\mu, \; E \subset \overline{x_0 z}. \tag{2.9}$$

For $E \subset M_z$, there exist two disjoint sets E_1 and E_2 in M_z such that

$$E = E_1 \cup E_2, \; E_1 \in M_0, \; E_2 \subset \overline{x_0 z}. \tag{2.10}$$

Hence for $E \in M_z$, we obtain from (2.9) and (2.10)

$$\| Tp(E) \|_m \leq \| q \|_n(E_1) + \int_{E_2} f|x, p(\overline{S_{xw}})|_m d\mu$$

which implies that

$$\| Tp \|_n \leq \| q \|_n + \delta.\varepsilon.\mu(\overline{x_0 z}) \leq b\varepsilon + (1-b)\varepsilon = \varepsilon,$$

since $\| q \| \leq b\varepsilon$, and $\delta = \frac{1-b}{\mu(x_0 z)}$.

This shows that T maps $\overline{S}(\varepsilon)$ into self. It can also be verified that, T is a contraction operator on $\overline{S}(\varepsilon)$. By an application of contraction mapping principle, there exists a unique solution $p(\overline{S_{x_0}}, q)$ of (1.1) and (1.2) satisfying $\| p \|_n < \varepsilon$, whenever $\| q \|_n \leq b\varepsilon$.

This completes the proof.

Remarks. (i) Different types of norms are used in this paper in order to meet certain requirements on the operator T defined in Theorems 2.1 and 2.4. Other convenient norms may also be used.

(ii) Theorems 2.1 and 2.3 can also be established when p and μ are complex vector measures, by defining S_x, $\overline{S_x}$ and B_H suitably.

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