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On Fixed Point Theorem Related to Banach's Contraction Principal

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Abstract

In this paper we used the nations of guasi-gauge space and of a Banach operator to consolidate the technique of the proof of Banach's contraction principle. Several examples are given to illustrate the results.

§ 1. Introduction

Calling a non-negative real function d on $X \times X$, (for a non-void set X) a quasi-pseudo metric on X provided d(x, x) = 0 for every x in X and $d(x,y) \leq d(x,y) + d(z,y)$ for all elements x,y,z in X .

Definition 1.1 : A quasi-gauge structure for space (X,T) is a family P of quasi-pseudo metrics on X such that T has as a subbase the family { B(x, p, ε) } : x in X, p in P, ε > 0 } where B (x, p, ε) is the set { y in X | p (x,y) < ε }. If a topological space (X, T) has a quasi-gauge structure P it is called a quasi-gauge space and is denoted by (X, P).

Remark : Every topological space is a quasi-gauge space. (See Theorem 2.6 [4]). If (X,d) is a metric space we take P to consist of d alone.

For our purpose we need the following concept of a Cauchy saquence due to Reilly [4] in a quasi-gauge space, generalizing the classical concept.

Definition 1.2 : If (X,P) is a quasi-gauge space then the squence $\{x_n\}$ in X is called left P-Cauchy. If for each p in P and each $\varepsilon > 0$ there is a point x in X and an integer k such that $p(x,x_m) < \varepsilon$ for all $m \geq k$. (x and k may depend on ε and p) .

More explicitly the sequence $\{x_n\}$ is a right P-Cauchy sequence if for each $\varepsilon > 0$ and p ε P there is an element x of X and an integer k such that $p(x_m,x) < \varepsilon$ for all $m \geq k$. The following example shows that a right P-Cauchy sequence need not be a left P-Cauchy sequence and vice versa.

Example 1.1: Let X be the interval (0.1) of real numbers and define the quasi-pseudo-metric p on X by $p(x,y) = \begin{cases} x-y & \text{if } x \geq y \\ 1 & \text{if } x < y \end{cases}$. Since for $x_n = 1/n$, n = 1, 2, ..., $p(x_n , x_n) < 1/n_0$ for all $n \geq n_0$ the sequence $x_n = 1/n$ is left P-Cauchy in X. However, this sequence is not right P-Cauchy as

$P(x_m,x) = 1$, for each x in X after a stage. On the other hand the sequence $\{x_n\} \equiv \{1-1/n\}^{\infty}_{n=2}$ is right P-Cauchy, as $P(x_m,X_n) < 1/n_0$ where $x_m = 1-1/m$ and $m \geq n_0$. But for each x in X, $p(x,x_m) = 1$, since $x_m > x$ after a certain stage. So $\{1-(1/n)\}^{\infty}_{n=2}$ is not a left P-Cauchy sequence.

Definition 1.3: A quasi-gauge space (X,P) is left (right) sequentially complete, if every left (right) P-Cauchy sequence in X converges to some element of X.

Definition 1.4 : An operator T on a quasi-gauge (X,P) is called a left Banach operator of type k if for each p in P there exists k (depending on p) such that $0 \leq k < 1$ and for x in X, $p(T(x), T^2(x)) \leq k\, p(x,T(x))$. T is called a right Banach operator of type k if for each p in P there exists k (depending on p) such that $0 \leq k < 1$ and for all x in X, $p(T^2(x), T(X)) \leq kp(T(X),x)$.

Remark : The operator $x \to x/2$ on the space introduced in example 1 is readily seen to be a left Banach operator, although it is not a right Banach operator.
In case P reduces to a single metric, we shall say that the Banach operator is of type k.

§ 2. Main Results

Theorem 2.1 : Let T be a continuous left (right) Banach operator on a Hausdorff left (right) sequentially complete quasi-gauge space (X,P) into itself. Then T has a fixed-point.

Proof : We shall prove the theorem in the case of a left Banach operator, omitting the case of the right Banach operator as the details are similar.

If x is in X, for each p in P it follows by induction that $p(T^n(x_0), T^{n+1}(x_0)) \leq k^n\, p(x_0, T(x_0))$. Let m and n be two positive integers such that m > n. Then

$$p\,(T^n(x_0), T^m(x_0)) \le \sum_{i=1}^{m-n} p\,(T^{n+i-1}(x_0), T^{n+i}(x_0))$$

$$\le \sum_{i=1}^{m-n} k^{n+i+1}\, p\,(x_0, T(x_0)).$$

As $0 \le k < 1$ for each p associated with p in P, $\sum_{i=1}^{n} k'$ converges and hence is a Cauchy sequence of real numbers. So given $\varepsilon > 0$ we can find n such that for all $m \ge n$, $\sum_{i=0}^{m-n} k^{n+i}\, p(x_0, T(x_0)) < \varepsilon$. So $p\,(T^n(_0), T^m(x_0)) < \varepsilon$ for all $m \ge n$. Thus for each p in P and $\varepsilon > 0$ we may choose x as $T^n(x_0)$ and see that $\{ T^n(x_0) \}$ is indeed a left P-Cauchy sequence in the sense of definition 2. Since (X,P) is left sequentially complete, $\{T^n(x_0)\}$ is convergent to some y in X. As T is a continuous operator, $\{T^{n+1}(x_0)\}$ is converges to T(y). $\{T^{n+1}(x_0)\}$ being a subsequence of $\{T^n(x_0)\}$ and X being a Hausdroff space, it follows that $T(y) = y$.

Remark Reilly's result ([4], Theorem 2) becomes a special case of Theorem 2.1 .

Theorem 2.2 : If T is a contractive operator on a quasi-gauge space (X,P) (i.e. for each $p \in P$ and some k (depending on p) with $0 \le k < 1$, $p(T(x), T(y)) \le k\, p\,(x,y)$ for all x, y in X), then T has a unique fixed-point in X, provided X is a left (right) sequentially complete Hausdroff space.

Proof : It can be easily seen that T is continuous and satisfies the hypothesis of theorem 2.1. So T has a fixed-point. Uniqueness of the fixed point is obvious, as X is Hausdroff.

Corollary 2.3 : If T be a continuous Banach operator on a complete metric space (X,d) then T has a fixed point .

Corollary 2.4 : If T be an operator on a complete metric space such that T^m is a continuous Banach operator with at most one fixed point, then T has a unique fixed-point.

Proof : By Corollary 2.3 of Theorem 2.1, T^m has a fixed-point y, which is unique (by our assumption). Since $y = T^m(y)$, $T(y) = T^{m+1}(y) = T^m(T)y))$, and the fixed-point of T is unique, it follows that $T(y) = y$.

Remarks : Corollary 2.4 fails if T has more than one fixed-point .

Example 2.2 : Let X be tha unit circle $\{(x,y) : x^2 + y^2 = 1\}$ with the usual Euclidean metric, and T be the map $(x,y) \rightarrow (-x,y)$. T^2 being the identify map is a continuous Banach operator. However T has no fixed point.

We have actually shown in Theorem 2.1 that every sequence of iterates of a continuous left (right) Banach operator on a left (right) sequentially complete quasi-gauge space converges to some fixed point. However a discontinuous Banach operator on a sequentially complete quasi-gauge space, even if it has a fixed point, need not have this property for the iterates. This is illustrated by the following example of a metric space.

Example 2.3 : Consider the set $(0) \cup \{(1/n) : n \text{ is a natural number}\} \cup (2)$ with the usual metric. Let T be the operator defined by $T(0) = 1$, $T(1/n) = (1/2n)$ for positive integral n and $T(2) = 2$. It is readily seen that $| T(x) - T^2(x) | \leq (3/4) | x - T(x) |$ for each element x. 2 is the only element whose iterates coverge to the fixed-point of T, which is evidently not a continuous operator.

The following example shows that a left sequentially complete quasi-gauge space need not be right sequentially complete.

Example 2.4 : Let X be the space (0,1) having the quasi-pseudo-metric p defined by

$$P(x,y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{if } x > y \end{cases}$$

The topology induced by p has as a base all the sets of the form [x,1], for x in X. X is not right sequentially complete since the right Cauchy sequence $\{1/n\}$ is not convergent in X . If $\{x_n\}$ is any left Cauchy sequence in X, then there is, by definition an element x in X such that for all $n \geq M$, $p(x,x_n) < ½$. From the definition of p. it now follows that for all $n \geq M$, $x_n \geq x$. This shows that $\{x_n\}$ converges to x, since it lies eventually in [x,1]. So X is left sequentially complete.

It may be noted that all sequences in the space of Example 4 are right Cauchy sequences.

For other results in this direction see [1, 2, 5] .

Theorem 2.5 : Let (X, P) be a Hausdroff sequentially complete quasi-gauge space generated by the family P of pseudometrics and T be an operator on X satisfying the following condition : for any x, y in X and for each p in P,

$$p\,(T(x),\,T(y) \le a\,p\,(x,T(x)) + b\,p\,(y,T(y)) + c\,p\,(x,T(Y)) + d\,p\,(y,T(x)) + e\,p(x,y) \qquad (2.1)$$

where a, b and e are real constants (varying with p) and c and d are non-negative numbers (varying with p). If for all p ∈ P,

either

$$(i)\ \ 1-b-c > 0 \ \text{ and } \ 0 \le \frac{a+c+e}{1-b-c} < 1$$

or

$$(ii)\ \ 1-a-d > 0 \ \text{ and } \ 0 \le \frac{b+d+e}{1-a-d} < 1$$

then T has a fixed-point. If in addition to either of these condition, for each p in P, c + d + e ≡ c(p) + d(p) + e(p) < 1 , then the fixed-point is unique.

Proof : Putting y ≡ T(x) in (2.1) and using p(x,T(x)) ≤ p(x,T(x)) + p(T(x), T^2(x)), we have for all x in X and p in P, as c and d are non-negative.

p (T(x), T^2(x)) ≤ a p (x,T(x)) + b p (T(x), T^2(x)) + c p (x,T^2(x)) + d p T(x),T^2(x)) + e p (x,T(x)) . If for all p ∈ P, 1 − b − c > 0 , then for each x in X and p ∈ P .

$$p\,(T(x),\,T^2(x)) \le \frac{a+c+e}{1-b-c}\ p(x,T(x)) \qquad (2.1.a)$$

Putting x = T(y) in (2.1), we get similarly, x in X and 1 > a + d,

$$p(T(x),\,T^2(x)) \le \frac{b+d+e}{1-a-d}\ p\,(x,T(x)) \qquad (2.1.b)$$

for all p in P . Therefore if T satisfies (i) or (ii) then T must be a Banach operator .

since (X,P) is sequentially complete and (T^n(x)) is a P-Cauchy sequence for each x in X, it converges to an element u of X.

For each natural number n > 1, p(u, T(u)) ≤ p(u,Tn(x)) for each pseudo-metric p ∈ P. If (2.1.a) is true, then putting x = T^{n-1}(x) and y = u in (2.1) and using triangle inequality, we get, for each p in P,

$p(T^n(x),\ T(u)) \le a\ p(T^{n-1}(x),\ T^n(x)) + b\ p(u,\ T(u)) + c[p(T^{n-1}(x),u) + p(u,T(u))] + d\ p(u,T^n(x)) + e\ p(T^{n-1}(x),u)$.

So

$$p(u,T(u)) \le \frac{1}{1-b-c}[(1+d)\,p(u,T^n(x)) + (c+e)p(T^{n-1}(x),u) + b\,p(T^{n-1}(x),T^n(x)] .$$

If (2.1.b) is true, then putting x = u and $y = T^{n-1}(x)$ in (2.1) and using triangle inequality, we similarly get for all p in P,

$$P(u,T(u)) \le \frac{1}{1-a-d}\ [(1+c)p(u,T^n(x)) + (d+e)p(T^{n-1}(x),u) + bp(T^{n-1}(x)\ T^n(x)]$$

since $[T^n(x)]$ converges to u and p $(T^{n-1}(x)\ ,\ T^n(x))$ tends to zero, it follows from the preceding inequalities that for each p in P, p(u,T(u)) = 0 . It now follows from the Hausdroof assumption that u = T(u).

If c + d + e < 1, then the fixed point of T is unique. For, if u and v are two fixed-points of T, then p(u,v) = p(T(u), T(v) ≤ (c+d+e) p(u,v), which follows from (2.1) on putting u = T(u) and v = T(v). Since c+d+e < 1, it follows that p(u,v) = 0 for all p in P. Since X is Hausdorff this in turn implies that u = v .

Remarks

1. Kannan [3] has provel the following result : Let T be an operator on a metric space (X,d) satisfying the condition that there exists a real number k with 0 < k < ½ such that for any x,y in X .

$$D(T(x),T(y)) \le k\ [d(x,T(x)) + d(y,T(y))] \qquad (2.2)$$

Then T has a unique fixed point, if X is complete. The above result follows from Theorem 2.5 by taking P = {d}, c = d = e = 0 and a = d with 0 < a+b < 1 .

2. In Theorem 2.5 all the real numbers a, b and e need not be positive as is evident from the following example :

Example 2.5 : Let T be the operator on [0,1] ∪ {– 4} (with the usual metric) defined by : T(– 4) and = – 4 and T (x) = 0 for each x in [0.1]. Using elementary inequalities it may he seen that T satisfies (2.1) with a = b = – ½ , c = d = e = ½ . Moreover $\frac{a+c+e}{1-b-c}$ = ½ and 1 – b – c > 0 .

3. Interchanging x and y in (2.1) and using the symmetry of each o the pseudo-metrics, (2.1) is reduced to the following : for x,y in X,

$$p(T(x),T(y)) \le \frac{a+b}{2}\,[p(x,T(x)) + p(y,(T(y))] + \frac{a+b}{2}\ [p(x,T(y)) + p(y,T(x))] + ep(x,y) \qquad (2.2)$$

The condition for T to be a Banach operator then becomes $2 \neq a+b+c+d$ and

$$0 < \frac{a+b+c+d+2e}{2-(a+b+c+d)} < 1$$

But this condition is more restrictive than what has been assumed in Theorem 2.5 . For instance the operator $x \rightarrow x^2$ on $[0,¼]$ satisfies (2.1) for $a = b = c = 0$, $d = 2$ and $e = ½$. Since $a + b + c + d = 2$, this criterion cannot be used to deduce that it is a Banach operator. But $\frac{a+c+e}{1-b-c} = ½$.

Now we have a stronger theorem, using the notion of a Banach operator .

Theorem 2.6 : Let (X.d) be a complete metric space and T be a continuous Banach operator having at most one fixed point and of type $k < ⅓$. Then T satisfies :

$$d(T(x),T(y)) \leq k\,[d(x,T(y)) + d(y,T(x))] \qquad (2.4)$$

Proof : For each x,y in X, it follows by induction that

$$d(T(x),T(y)) \leq \sum_{i=1}^{n-1} [d(T^i(x),T^{i+1}(x)) + d(T^i(y),T^{i+1}(y))] + d(T^n(x),T^n(y))$$

Since T is a Banach operator of type k, it follows that

$$d(T(x),T(y)) \leq \sum_{i=1}^{n} k^i\,[d(x,T(x) + d(y,T(y))] + d(T^n(x),T^n(y)) \qquad (2.5)$$

By Theorem 2.3, T has a fixed point which is unique by assumption. Moreover $d(T^n(x),T^n(y))$ tends to zero as n tends to ∞, since each sequence of iterates converges to the fixed-point. So proceeding to the limit in (2.5) and noting that $k < 1$ we have

$$d(T(x), T(y)) \leq \sum_{i=1}^{\infty} k^i\,[d(x,T(x)) + d(y,T(y))]$$

$$\leq k\,[d(x,T(x)) + d(y,T(y))]/(1-k)$$

Since $k < ⅓$, it follows that $k/(1-k) < ½$ and so T satisfies (2.2).

Corollary 2.7 : If T is a continuous Banach operator of type k ($< ⅕$) and having at most one fixed point in a complete metric space (X,d) then T satisfies (2.4)

Proof : From Theorem 2.6 it follows that T satisfies (2.2) for the constant $k' = k/(1-k)$. Since $k < ⅕$, it follows that $k' < ¼$. Hence by the discussion preceding Theorem 2.6, T satisfies (2.4) also for $k'' = k'/(1-2k') = k/(1-3k)$.

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